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Flavor Oscillations from a Spatially Localized Source A Simple General Treatment

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Abstract

A unique description avoiding confusion is presented for all flavor oscillation experiments in which particles of a definite flavor are emitted from a localized source. The probability for finding a particle with the wrong flavor must vanish at the position of the source for all times. This condition requires flavor–time and flavor–energy factorizations which determine uniquely the flavor mixture observed at a detector in the oscillation region; i.e. where the overlaps between the wave packets for different mass eigenstates are almost complete. Oscillation periods calculated for "gedanken" time-measurement experiments are shown to give the correct measured oscillation wave length in space when multiplied by the group velocity. Examples of neutrinos propagation in a weak field and in a gravitational field are given. In these cases the relative phase is modified differently for measurements in space and time. Energy-momentum (frequency-wave number) and space-time descriptions are complementary, equally valid and give the same results. The two identical phase shifts obtained describe the same physics; adding them together to get a factor of two is double counting.

I. INTRODUCTION

Flavor oscillations are observed when a source creates a particle which is a mixture of two or more mass eigenstates, and a different mixture is observed in a detector. Such oscillations have been observed in the neutral kaon and B–meson systems. In neutrino experiments it is still unclear whether the eigenstates indeed have different masses and whether oscillations can be observed. Considerable confusion has arisen in the description of such experiments in quantum mechanics [1,2], with questions arising about time dependence and production reactions [3], and defining precisely what is observed in an experiment [4]. Many calculations describe "gedanken" experiments and require some recipe for applying the results to a real experiment [5].

We resolve this confusion by noting and applying one simple general feature of all practical experiments. The size of the source is small in comparison with the oscillation wave length to be measured, and a unique well–defined flavor mixture is emitted by the source; e.g. electron neutrinos in a neutrino oscillation experiment. The particles emitted from the source must therefore be described by a wave packet which satisfies a simple general boundary condition: the probability amplitude for finding a particle having the wrong flavor at the source must vanish at all times.

This boundary condition requires factorization of the flavor and time dependence at the position of the source. Since the energy dependence is the Fourier transform of the time dependence, this factorization also implies that the flavor dependence of the wave packet is independent of energy at the position of the source. In a realistic oscillation experiment the relative phase is important when the oscillation length is of the same order as the distance between the source and the detector. In that case this flavor–energy factorization holds over the entire distance between the source and detector. The boundary condition then determines the relative phase of components in the wave function with different mass having the same energy and different momenta. Thus any flavor oscillations observed as a function of the distance between the source and the detector are described by considering

only the interference between a given set of states having the same energy. All questions of coherence, relative phases of components in the wave function with different energies and possible entanglements with other degrees of freedom are thus avoided.

Many formulations describe flavor oscillations in time produced by interference between states with equal momenta and different energies. These "gedanken" experiments have flavor oscillations in time over all space including the source. We show rigorously that the ratio of the wave length of the real spatial oscillation to the period of the gedanken time oscillation is just the group velocity of the wave packet.

II. UNIVERSAL BOUNDARY CONDITION

We now show how the results of a flavor oscillation experiment are completely determined by the propagation dynamics and the boundary condition that the probability of observing a particle of the wrong flavor at the position of the source at any time must vanish. We choose for example a neutrino oscillation experiment with a source of neutrinos of a given flavor, say electron neutrinos*. The dimensions of the source are sufficiently small in comparison with the distance to the detector so that it can be considered a point source at the origin. The neutrino wave function for this experiment may be a very complicated wave packet, but a sufficient condition for our analysis is to require it to describe a pure ν_e source at $x = 0$; i.e. the probability of finding a ν_μ or ν_τ at $x = 0$ is zero.

We first consider propagation in free space, where the masses and momenta p_i satisfy the usual condition

$$p_i^2 = E^2 - m_i^2 \,. \tag{2.1}$$

We expand the neutrino wave function in energy eigenstates

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*For simplicity, we do not consider possible effects of physics beyond the Standard Model on neutrino interactions [6]. The generalization to this case is straightforward.

$$\psi = \int g(E) dE e^{-iEt} \cdot \sum_{i=1}^{3} c_i e^{ip_i \cdot x} \left| \nu_i \right\rangle \;, \tag{2.2}$$

where $|\nu_i\rangle$ denote the three neutrino mass eigenstates and the coefficients c_i are energy-independent. Each energy eigenstate has three terms, one for each mass eigenstate. In order to avoid spurious flavor oscillations at the source the particular linear combination of these three terms required to describe this experiment must be a pure ν_e state at $x = 0$ for each individual energy component. Thus the coefficients c_i satisfy the conditions

$$\sum_{i=1}^{3} c_i \left\langle \nu_i \right| \nu_\mu \rangle = \sum_{i=1}^{3} c_i \left\langle \nu_i \right| \nu_\tau \rangle = 0 \,. \tag{2.3}$$

The momentum of each of the three components is determined by the energy and the neutrino masses. The propagation of this energy eigenstate, the relative phases of its three mass components and its flavor mixture at the detector are completely determined by the energy-momentum kinematics for the three mass eigenstates.

The exact form of the energy wave packet described by the function $g(E)$ is irrelevant at this stage. The components with different energies may be coherent or incoherent, and they may be "entangled" with other degrees of freedom of the system. For the case where a neutrino is produced together with an electron in a weak decay the function $g(E)$ can also be a function $g(\vec{p}_e, E)$ of the electron momentum as well as the neutrino energy. The neutrino degrees of freedom observed at the detector will then be described by a density matrix after the electron degrees of freedom have been properly integrated out, taking into account any measurements on the electron. However, none of these considerations can introduce a neutrino of the wrong flavor at the position of the source.

Since the momenta p_i are energy-dependent the factorization does not hold at finite distance. At very large values of x the wave packet must separate into individual wave packets with different masses traveling with different velocities [7,1]. However, for the conditions of a realistic oscillation experiment this separation has barely begun and the overlap of the wave packets with different masses is essentially 100%. Under these conditions the flavor–energy factorization introduced at the source is still an excellent approximation at the detector.

The flavor mixture at the detector given by substituting the detector coordinate into Eq. (2.2) can be shown to be the same for all the energy eigenstates except for completely negligible small differences. For example, for the case of two neutrinos with energy E and mass eigenstates m_1 and m_2 the relative phase of the two neutrino waves at a distance x is:

$$\delta\phi(x) = (p_1 - p_2) \cdot x = \frac{(p_1^2 - p_2^2)}{(p_1 + p_2)} \cdot x = \frac{\Delta m^2}{(p_1 + p_2)} \cdot x \,, \tag{2.4}$$

where $\Delta m^2 \equiv m_2^2 - m_1^2$. Since the neutrino mass difference is very small compared to all neutrino momenta and energies, we use $|m_2 - m_1| \ll p \equiv (1/2)(p_1 + p_2)$. Thus we can rewrite Eq. (2.4) keeping terms only of first order in Δm^2

$$\delta\phi(x) = \frac{\Delta m^2}{2p} \cdot x = -\left(\frac{\partial p}{\partial(m^2)}\right)_E \Delta m^2 \cdot x \,, \tag{2.5}$$

where the standard relativistic energy-momentum relation (2.1) gives the change in energy or momentum with mass when the other is fixed,

$$\left(\frac{2E\partial E}{\partial(m^2)}\right)_p = -\left(\frac{2p\partial p}{\partial(m^2)}\right)_E = 1 \,. \tag{2.6}$$

Thus we have a complete solution to the oscillation problem and can give the neutrino flavor as a function of the distance to the detector by examining the behavior of a single energy eigenstate. The flavor–energy factorization enables the result to be obtained without considering any interference effects between different energy eigenstates. The only information needed to predict the neutrino oscillations is the behavior of a linear combination of the three mass eigenstates having the same energy and different momenta. All effects of interference or relative phase between components of the wave function with different energies are time dependent and are required to vanish at the source, where the flavor is time independent. This time independence also holds at the detector as long as there is significant overlap between the wave packets for different mass states. The conditions for the validity of this overlap condition are discussed below.

Neutrino states with the same energy and different momenta are relevant rather than vice versa because the measurement is in space, not time, and flavor–time factorization holds in a definite region in space.

III. RELATION BETWEEN REAL AND GEDANKEN EXPERIMENTS

We now derive the relation between our result (2.4) which comes from interference between states with the same energy and different momenta and the standard treatments using states with the same momentum and different energies [8]. For the case of two neutrinos with momentum p and mass eigenstates m_1 and m_2 the relative phase of the two neutrino waves at a time t is:

$$\delta\phi(t) = (E_2 - E_1) \cdot t = \left(\frac{\partial E}{\partial (m^2)} \right)_p \Delta m^2 \cdot t = - \left(\frac{\partial p}{\partial (m^2)} \right)_E \Delta m^2 \cdot \frac{p}{E} \cdot t \,, \tag{3.1}$$

where we have substituted Eq. (2.6). This is equal to the result (2.5) if we make the commonly used substitution

$$x = \frac{p}{E} \cdot t = vt \,. \tag{3.2}$$

This is now easily generalized to include cases where external fields can modify the relation (2.1), but where the mass eigenstates are not mixed. The extension to propagation in a medium which mixes mass eigenstates e.g. by the MSW effect [9] is in principle the same, but more complicated in practice and not considered here. The relation between energy, momentum and mass is described by an arbitrary dispersion relation

$$f(E, p, m^2) = 0 \,, \tag{3.3}$$

where the function f can also be a slowly varying function of the distance x. In that case, the momentum p for fixed E is also a slowly varying function of x. We take this into account by expressing Eq. (2.5) as a differential equation, and defining the velocity v by the conventional expression for the group velocity,

$$\frac{\partial^2 \phi(x)}{\partial x \partial (m^2)} = - \left(\frac{\partial p}{\partial (m^2)} \right)_E = \frac{1}{v} \cdot \left(\frac{\partial E}{\partial (m^2)} \right)_p \,, \qquad v \equiv \left(\frac{\partial E}{\partial p} \right)_{(m^2)} \,. \tag{3.4}$$

Treatments describing real experiments measuring distances and "gedanken" experiments measuring time are seen to be rigorously equivalent if the group velocity (3.4) relates the two results. Note that the group velocity and not the phase velocity enters into this relation.

The relations (3.4) are trivial and obvious for the case of neutrinos propagating in free space, and gives Eq. (3.2). However, it becomes nontrivial for more complicated cases. Two such cases are presented in the following.

IV. DESCRIPTION IN TERMS OF TIME BEHAVIOR

The specific form of the wave packet given by the function $g(E)$ in Eq. (2.2) describes the Fourier transform of the time behavior as seen at $x = 0$. This time behavior changes as the packet moves from source to detector. Components corresponding to different mass eigenstates move with different velocities. When the centers of the wave packets have moved a distance x_c they have separated by a distance

$$\delta x_c = \frac{\delta v}{v} \cdot x \approx \frac{\delta p}{p} \cdot x = \frac{\Delta m^2}{2p^2} \cdot x \,, \qquad \delta v \equiv v_1 - v_2 \,, \qquad \delta p \equiv p_1 - p_2 \,, \tag{4.1}$$

where v_1, v_2 and v denote the individual group velocities of the two wave packets and an average group velocity, and we have assumed that $m_i^2 = E_i^2 - p_i^2 \ll p_i^2$. This separation between the wave packet centers produces a phase displacement between the waves at the detector, $\delta\phi(x) = p\,\delta x_c$, which is seen to give exactly the same phase shift as Eq. (2.4). The group velocity which determines the separation between the wave packets is relevant and not the phase velocity.

Further insight into the relation between different treatments is seen by rewriting the phase shift Eq. (2.4) in terms of the distance $\xi \equiv x - x_c$ between the point x and the center of the wave packet as the sum of the relative phase shift between the centers of the two wave packets $\delta\phi(x_c)$ at a fixed time and a "correction" to this phase shift because the centers of the wave packets arrive at the detector at different times. To first order in the small quantities δx and δp

$$\delta x_c + \delta\xi = 0 \,, \qquad \delta\phi(x) = \delta(xp) = x\delta p + p\delta x_c + p\delta\xi = \delta\phi(x_c) + p\delta\xi \,, \tag{4.2}$$

$$\delta\phi(x_c) \equiv x\delta p + p\delta x_c = \frac{\Delta m^2}{p} \cdot x \,, \qquad p\delta\xi = -p\delta x_c = -\frac{\Delta m^2}{2p} \cdot x \,. \tag{4.3}$$

Writing the phase shift in this form and neglecting the "correction" leads to an overestimate of the phase by a factor of two, while adding the "correction" to the correct interpretation (3.1) of the gedanken experiment can lead to double counting.

We see here simply another description of the same physics used in the derivation of Eq. (2.4), using the complementarity of energy-momentum and space-time formulations. They are two ways of getting the same answer, not two different effects that must be added.

The same complementarity is seen in the interference between two classical wave packets moving with slightly different velocities. Even without using the quantum mechanical relations with energy and momentum there are two possible descriptions, one using space and time variables and one using frequency and wave length. The two descriptions are Fourier transforms of one another and give the same result. Adding the two results is double counting.

We now apply this picture of two wave packets traveling with slightly different velocities to examine the time-dependent probability amplitude for a neutrino wave seen at the detector when it is emitted from the source in a flavor eigenstate denoted by $|f_1\rangle$. The x dependences of the amplitude and other parameters are suppressed since we only need their values at the position of the detector.

$$|\Psi(t)\rangle = e^{i\phi_o(t)} \left[\cos\theta A(t)\, |m_1\rangle + \sin\theta A(t+\tau) e^{i\phi(\tau)}\, |m_2\rangle\right] , \tag{4.4}$$

where $|m_1\rangle$ and $|m_2\rangle$ denote the two mass eigenstates and θ is a mixing angle defining the flavor eigenstates denoted by $|f_1\rangle$ and $|f_2\rangle$ in terms of the mass eigenstates,

$$|f_1\rangle = \cos\theta\, |m_1\rangle + \sin\theta\, |m_2\rangle \ , \qquad |f_2\rangle = \sin\theta\, |m_1\rangle - \cos\theta\, |m_2\rangle \ , \tag{4.5}$$

$$\tau(x) = \frac{x}{v_2} - \frac{x}{v_1} \approx \frac{\delta v}{v^2} x \approx \frac{\Delta m^2}{2p^2\, v} x \, , \tag{4.6}$$

where $\delta v/v$ is always defined for components in the different mass eigenstates having the same energy and the small variation in $\delta v/v$ over the wave packet is neglected. We express each mass eigenstate wave function as the product of a magnitude $A(x)$ and a phase. The

universal boundary condition requires A to be the same for both mass eigenstates at the source. The wave functions spread with distance and may become much broader at the detector. However the difference in shape between the two mass eigenstates is shown below to be negligible at the detector under experimental conditions where oscillations are observable. Their center difference is described by the time displacement τ.

The probability amplitudes for observing the flavor eigenstates at the detector are

$$\langle f_1 \,|\Psi(t)\rangle = e^{i\phi_o(t)} \left[\cos^2\theta A(t) e^{i\phi(\tau)} + \sin^2\theta A(t+\tau)\right] , \tag{4.7}$$

$$\langle f_2 \,|\Psi(t)\rangle = e^{i\phi_o(t)} \sin\theta\cos\theta \left[A(t) e^{i\phi(\tau)} - A(t+\tau)\right] . \tag{4.8}$$

The relative probabilities that flavors f_1 and f_2 are observed at the detector are

$$P(f_1,\tau) = \int dt |\langle f_1 \,|\Psi(t)\rangle\,|^2 = 1 - \frac{\sin^2(2\theta)}{2}\Big[1 - O(\tau)\cos\phi(\tau)\Big] , \tag{4.9}$$

$$P(f_2,\tau) = \int dt |\langle f_2 \,|\Psi(t)\rangle\,|^2 = \frac{\sin^2(2\theta)}{2}\Big[1 - O(\tau)\cos\phi(\tau)\Big] , \tag{4.10}$$

where the amplitude normalization and the overlap function $O(\tau)$ are given by

$$\int dt |A(t)|^2 = 1 , \qquad O(\tau) \equiv \int dt A(t+\tau) A(t) . \tag{4.11}$$

When the overlap is complete, $O(\tau) \approx 1$, the results (4.9) and (4.10) reduce to the known result obtained by assuming plane waves [8] and using

$$\phi(\tau) = p\delta x_c = p\, v\, \tau \approx \frac{\Delta m^2}{2p} x . \tag{4.12}$$

An explicit example for the calculation of the overlap function can be found in Ref. [10] where the shape function A was taken to be a Gaussian.

We now examine the spreading of the wave functions while traveling from the source to the detector. The length of the wave packet in space $L_w(0)$ in the vicinity of the source must be sufficiently large to contain a large number N_w of wave lengths $\lambda\!\!\!{}^-$ in order to define a phase. This then determines the spread of the momentum, δp_w, and velocity, δv_w, in the wave packet

$$L_w(0) = N_w \lambda\!\!\!^{-} = \frac{N_w}{p} \,, \qquad \frac{\delta p_w}{p} = \frac{\delta v_w}{v} = \frac{1}{N_w} \,. \tag{4.13}$$

The spreading of the wave packet in traveling from the source to the point x is

$$\frac{L_w(x) - L_w(0)}{L_w(0)} = \frac{\delta v_w}{v} \cdot \frac{x}{L_w(0)} = \frac{\delta p_w}{p} \cdot \frac{x \cdot p}{N_w} = \frac{x \cdot p}{N_w^2} \,. \tag{4.14}$$

The difference in the spreading of the wave packets for the different mass eigenstates is then seen to be negligible for distances x where the oscillation phase shift $\delta\phi(x)$ is of order unity

$$\frac{\partial}{\partial(m^2)} \left(\frac{L_w(x) - L_w(0)}{L_w(0)} \right) \cdot \Delta m^2 = \frac{\partial p}{\partial(m^2)} \cdot \Delta m^2 \cdot \frac{x}{N_w^2} = \frac{\delta\phi(x)}{N_w^2} \,. \tag{4.15}$$

The different mass eigenstates separate as a result of the velocity differences. Eventually the wave packet separates into distinct packets, one for each mass, moving with different velocities. The separation destroys the flavor–energy and flavor–time factorizations and introduces a time dependence in the flavor observable in principle at a given large distance. In practice the detailed time dependence is not measurable and only the attenuation of the oscillation expressed by the overlap function $O(\tau)$ is seen. When the wave packets for different masses no longer overlap there is no longer any coherence and there are no further oscillations [7]. The result (2.4) applies for the case where the separation (4.1) is small compared to the length in space of the wave packet; i.e. when the eventual separation of the wave packets has barely begun and can be neglected.

V. FUZZINESS IN TIME

The oscillations can be described either in space or in time. But the distance between the source and the detector is known in a realistic experiment to much higher accuracy then the time interval. Thus the interval between the two events of creation and detection has a sharp distance and a fuzzy time in the laboratory system. A Lorentz transformation to a different frame necessarily mixes distance and time and makes both fuzzy in a complicated manner. For this reason one must be careful in interpreting any results obtained in other

frames than the laboratory system. The proper time interval between the two events is always fuzzy.

The fuzziness of the time is an essential feature of the experiment since the wave packet has a finite length L_w in space. The probability of observing the particle at the detector is spread over the time interval

$$2\delta t \equiv \frac{L_w}{v} = \frac{L_w E}{p}\,. \tag{5.1}$$

The proper time interval τ between emission and detection is given by

$$\tau^2 = (t \pm \delta t)^2 - x^2 = x^2 \left[\frac{m^2}{p^2} + \frac{L_w^2 E^2}{4x^2 p^2} \pm \frac{L_w E^2}{x p^2}\right] = x^2\, \frac{m^2}{p^2} \left[1 + \frac{E^2}{m^2} \cdot \left(\frac{L_w^2}{4x^2} \pm \frac{L_w}{x}\right)\right]. \tag{5.2}$$

This uncertainty in the proper time interval due to the finite length of the wave packet cannot be neglected.

The waves describing the propagation of different mass eigenstates can be coherent at the detector only if the overlap function $O(\tau)$ given by Eq. (4.11) is nearly unity. Thus the time interval between creation and detection is not precisely determined and subject to quantum–mechanical fluctuations. The length L_w of the wave packet created at the source must be sufficiently long to prevent the determination of its velocity by a time measurement with the precision needed to identify the mass eigenstate.

The small dimensions of the source introduce a momentum uncertainty essential for the coherence of the waves of different mass eigenstates. The wave packet describing the experiment must necessarily contain components from different mass eigenstates with the same energy and different momenta.

Conventional experiments measure distances to a precision with an error tiny in comparison with the oscillation wave length to be measured. This is easily achieved in the laboratory. In a "gedanken" experiment where oscillations in time are measured, the experimental apparatus must measure times to a precision with an error tiny in comparison with the oscillation period to be measured. One might envision an experiment which measures the time the oscillating particle is created by observing another particle emitted at the same

time; e.g. an electron emitted in a beta decay together with the neutrino whose oscillation is observed. But if both the time and position of the created particle are measured with sufficient precision a very sharp wave packet is created and the mass eigenstates moving with different velocities quickly separate, the overlap function $O(\tau)$ approaches zero and there is no coherence and no oscillation.

In reality, when both x and t are measured there are fluctuations in their values. Using $v = x/t$ the fluctuations in x and t must be large enough to make the velocity fuzzy. Then, in order to have oscillation we need the fuzziness in velocity to be much larger than the difference between the two group velocities, $\delta v_w \gg \delta v$. This is the case in a real experiment. Typical values are [11] $E = O(10\ MeV)$; $x = O(10^2\ m)$; $t = O(10^{-6}\ sec)$ and the relevant masses that can be probed are $\Delta m^2 = O(1\ eV^2)$. Then, $\delta v = O(10^{-12})$. Since $\delta v_w \approx dx/x + dt/t$ we see that the accuracies needed to measure the separate velocities are $dx = O(10^{-10}\ m)$ and $dt = O(10^{-18}\ sec)$, far from the ability of present technology. This calculation can also be performed for all terrestrial experiments, finding that the present technology is not yet sufficiently precise to destroy coherence and prevent oscillations from being observed.

VI. EXAMPLES

The relations (3.4) are trivial and obvious for the case of neutrinos propagating in free space. However, it becomes nontrivial for more complicated cases. In this section we present two nontrivial examples: Neutrino in a (flavor blind) weak field and neutrino in a gravitational field. These are only examples, in real life the effects we discuss tend to be very small, and consequently negligible. Yet, these examples demonstrate how to get the phase shift, and how to move from the description in terms of time to that of space using the group velocity.

In these examples we calculate the phase difference for a known beam with known energy. We consider a source and a detector in vacuum and investigate the effect of inserting a field

(either weak or gravitational) between them.

A. Neutrino in a weak field

We consider neutrino travel in a flavor–blind medium. The medium changes the dispersion relation [9] by introducing the potential V describing the scattering in the medium

$$(E+V)^2 - p^2 = m^2 \,. \tag{6.1}$$

For simplicity we assume that V is independent of x but can depend upon E. The phase difference in space and in time are then given by

$$\delta\phi(x) = -\left(\frac{\partial p}{\partial(m^2)}\right)_E \Delta m^2 \cdot x = \frac{\Delta m^2}{2p} \cdot x \approx \frac{\Delta m^2}{2p_o} (1-\epsilon) \cdot x \,, \tag{6.2}$$

$$\delta\phi(t) = -\left(\frac{\partial E}{\partial(m^2)}\right)_p \Delta m^2 \cdot t = \frac{\Delta m^2}{2(E+V)(1+\frac{dV}{dE})} \cdot t \approx \frac{\Delta m^2}{2E} \, \frac{1-\epsilon}{1+\epsilon'} \cdot t \,, \tag{6.3}$$

where $p \approx E+V$ and $p_o \approx E$ are the momentum in the medium and in free space, respectively. We work to first order in ϵ and ϵ' defined as

$$\epsilon \equiv \frac{V}{E} \,, \qquad \epsilon' = \frac{dV}{dE} \,. \tag{6.4}$$

We learn that the medium effect is *different* for the two cases

$$\frac{\delta\phi(x)}{\delta\phi_o(x)} = 1-\epsilon \,, \qquad \frac{\delta\phi(t)}{\delta\phi_o(t)} = \frac{1-\epsilon}{1+\epsilon'} \,, \tag{6.5}$$

where $\delta\phi_o(x)$ and $\delta\phi_o(t)$ denote the values respectively of $\delta\phi(x)$ and $\delta\phi(t)$ for the case where $V=0$. To move from one description to the other we need the group velocity

$$v = \left(\frac{\partial E}{\partial p}\right)_{(m^2)} = \frac{p}{(E+V)(1+\frac{dV}{dE})} = \frac{1}{1+\epsilon'} \,. \tag{6.6}$$

Using $t \to x/v = x(1+\epsilon')$ in (6.3) we get (6.2). We see that by using the correct velocity one can relate the two descriptions and the results are the same.

Note that our example is not realistic. In the Standard Model the neutral current interactions (that are flavor blind) are energy independent. Then, $\epsilon' = 0$ and the group velocity is not changed from its vacuum value.

This example has a simple optical analog. Consider an optical interference experiment (e.g. a two slit experiment) with a glass inserted in the light path. A measurement in space will gain a larger phase shift due to the travel in the medium. The light travels slower in the medium and when it reaches the detector the optical path is longer.

B. Neutrino in a gravitation field

We consider neutrino travel in a gravitational field. This has recently been treated in Refs. [12–14]. We compare two cases: one when the neutrino travel is in free space, a second when a gravitational field is inserted in the path. We assume that the gravitational field is sufficiently small to leave the (Newtonian) distance unaffected by its insertion. One example is the possible effect of the moon on solar neutrinos when the moon is close to solar eclipse. Then we shall see that the gravitational field of the moon affects the phase.

We assume: (1) The semi–classical limit; (2) The weak field limit; (3) Nearly Newtonian gravitational fields. The first assumption [15] says that gravity is not quantized and its effect is introduced by a nonflat space-time metric $g_{\mu\nu} \neq \eta_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the flat metric. The second assumption [16] says that we can use the linear approximation. Then, gravity is treated as an external field on a flat space time and we expand

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{6.7}$$

with $|h_{\mu\nu}| \ll 1$. The third assumption [16] says that the gravitational field originates from a massive static source. Then

$$h_{\mu\mu} = 2\Phi(\vec{x}) \,, \qquad h_{\mu\nu} = 0 \;\; \text{for} \;\; \mu \neq \nu \,, \tag{6.8}$$

where $\Phi(\vec{x})$ is the Newtonian potential (e.g. $\Phi(\vec{x}) = -G\, M/|\vec{x}|$ for a spherically symmetric

object with mass M). We emphasize that $h_{00} = h_{ii}$ but $\eta_{00} = -\eta_{ii}$. This sign difference turns out to be important.

The dispersion relation in a curved space-time is [16]

$$g_{\mu\nu} p^\mu p^\nu = m^2 \,, \tag{6.9}$$

where $p^\mu = m dx^\mu / ds$ is the local momentum, and ds is the distance element of general relativity: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. We consider neutrinos that travel in space-time from A to B. The wave function is then [15]

$$\psi = \exp(i\phi), \qquad \phi = \int_A^B g_{\mu\nu} p^\mu dx^\nu \,. \tag{6.10}$$

The phase difference in space and in time are then given by

$$\delta\phi(x) = \int_A^B g_{11}(p_2 - p_1) dx = \int_A^B \left(\frac{g_{11} \partial p}{\partial (m^2)} \right)_E \Delta m^2 dx \,, \tag{6.11}$$

$$\delta\phi(t) = \int_A^B g_{00}(E_2 - E_1) dt = \int_A^B \left(\frac{g_{00} \partial E}{\partial (m^2)} \right)_p \Delta m^2 dt \,. \tag{6.12}$$

The velocity is then obtained by generalizing Eq. (3.4)

$$v = -\left(\frac{g_{00} \partial E}{g_{11} \partial p} \right)_{(m^2)} \,. \tag{6.13}$$

Applying this to the dispersion relation we get

$$\delta\phi(x) = \int_A^B \frac{\Delta m^2}{2p} dx \approx \int_A^B (1 - \Phi(\vec{x})) \frac{\Delta m^2}{2p_o} dx \,, \tag{6.14}$$

$$\delta\phi(t) = \int_A^B \frac{\Delta m^2}{2E} dt \approx \int_A^B (1 + \Phi(\vec{x})) \frac{\Delta m^2}{2E_o} dt \,, \tag{6.15}$$

where $p_o^\mu = m dx^\mu / ds_o$ is the usual momentum of special relativity (global momentum) [15]. We work to first order in $\Phi(\vec{x})$ and we use [16,15]

$$p \approx p_o(1 + \Phi(\vec{x})) \,, \qquad E \approx E_o(1 - \Phi(\vec{x})) \,. \tag{6.16}$$

Our result (6.14) is the one obtained in [12].

We learn that the gravitational effect is *different* for the two cases

$$\frac{\delta\phi(x)}{\delta\phi_o(x)} = \frac{\lambda_o}{\lambda} = 1 - \epsilon\,, \qquad \frac{\delta\phi(t)}{\delta\phi_o(t)} = \frac{\tau_o}{\tau} = 1 + \epsilon\,, \tag{6.17}$$

where λ and λ_o denote the wave length of the oscillation in space for the case with and without the gravitational field respectively and similarly τ and τ_o denote the period of the oscillation in time for the two cases and we define

$$\epsilon \equiv \int_A^B \Phi(\vec{x})\frac{\Delta m^2}{2p_o}dx \approx \int_A^B \Phi(\vec{x}(t))\frac{\Delta m^2}{2E_o}dt\,. \tag{6.18}$$

Note that the effect of the gravitational field on the oscillation wave length λ in space is exactly opposite to the effect on the oscillation period τ in time. In order to move from one description to the other we need the velocity. From (6.13) we get

$$v = \frac{p}{E} \approx 1 + 2\Phi(\vec{x})\,, \tag{6.19}$$

which is the known result of the speed of light in a gravitational field [16]. Using $t \to x/v \approx x(1 - 2\Phi(\vec{x}))$ in (6.15) we get (6.14).

It is important to understand the meaning of this shift. We work in the example given before, and examine the effect of moon gravity on solar neutrinos. Since we assume that the earth–sun distance is not changed the effect can be viewed in two equivalent ways. One is the point of view of the linearized theory of gravity [16]. Then, space-time is flat and gravity is treated as a tensor field. In this approach, taken by [12], the neutrino travels the same distance with and without the moon, but gravity slows down the neutrino, thus it has a longer "optical" path and a larger phase is acquired. The second point of view is to work within the framework of general relativity. Then gravity is treated by changing the metric into curved space-time. In this approach, taken by [13], the neutrino always travels in free space. However, when the moon comes close to the sun-earth line the distance the neutrino has to travel is larger. The effect of gravity is then moved into the boundary of the integral, and we see that a larger phase is acquired. Of course, if one compares two experimental setups with and without gravity with the same curved distance in both cases there is no effect [13].

The analog of the two points of view is the famous “bending of light”. When light travels near the sun it is bent. This can be understood in two equivalent ways. Either that gravity acts on the light and curves its path, or that the space near the sun is curved. With either point of view, the final result is the same, we observe the bending of the light.

It is instructive to see how the effect can be obtained from the description in terms of time behavior. Then we just need the distance between the centers of the wave packets (4.1), or equivalently, the time between their arrivals. This time difference can be calculated by taking two classical relativistic particles with the same energy and different masses leaving the source. Then, the time difference of their arrival can be calculated. The result shows the gravitational effect. The time delay is sensitive to the presence of the gravitational field in the path.

Finally, we comment about the interplay between the gravitational and the MSW effects. In order for the gravitational effect to be appreciable a very strong gravitational field must be present. This may be the case in supernova. In this case there is also a weak field originating from the matter in the star, or from the neutrinos themselves [9]. In general, this tends to significantly reduce the mixing angles [17] very near to the value zero in which the flavor eigenstate ν_e is also a mass eigenstate. In the adiabatic limit a neutrino created in matter in a mass eigenstate remains a single mass eigenstate throughout its career. Its flavor can flip in a manner that explains the solar neutrino puzzle [8], but there are no oscillations and the gravitational phase cannot be observed. Of course gravity effects can be important beyond the effect on the coherent phase. We do not study such effects here.

VII. CONCLUSIONS

The complete description of a flavor oscillation experiment requires knowledge of the density matrix for the flavor-mixed state. This depends upon the production mechanism and possible entanglements with other degrees of freedom as well as on other dynamical factors which are often ignored. A proton in a fixed-target experiment is not really free

but bound by some kind of effective potential with characteristic lattice energies like Debye temperatures, which are of the order of tens of millivolts. This energy scale is no longer negligible in comparison with mass differences between flavor eigenstates [18]. The bound proton is not strictly on shell and has potential as well as kinetic energy. Arguments of Galilean and Lorentz invariance and separation of center-of-mass motion may not hold for the kinematics of the production process if the degrees of freedom producing the binding are neglected.

In this paper all these complications are avoided and a unique prescription has been given for the relative phases of the contributions from different mass eigenstates to a flavor oscillation experiment with a localized source having a well defined flavor. The boundary condition that the probability of observing a particle of the wrong flavor at the source position must vanish for all times requires a factorization in flavor and energy of the wave function at the position of the source. This uniquely determines the wave length of the oscillations observed at the detector as long as the overlap between wave packets for different mass eigenstates is maintained at the position of the detector.

Whether this wave-packet overlap is sufficiently close to 100% at the detector depends upon other parameters in the experiment which determine the detailed time behavior of the wave packet. If this overlap is appreciable but no longer nearly complete, the time behavior of the flavor mixture at the detector can be extremely complicated with leading and trailing edges of the wave packet being pure mass eigenstates and the intermediate region having a changing flavor mixture depending upon the relative magnitudes of the contributing mass eigenstates as well as the relative phases. This detailed behavior is not observable in practice; only the time integral is measured.

A unique prescription has been given for interpreting results of calculations for "gedanken" experiments which measure oscillations in time for components in the wave packets having the same momentum and different energies. The period of oscillation in time is related to the wave length of oscillation in space by the group velocity of the waves.

Results are simple in the laboratory system where the positions of the source and detector

are sharp in comparison with all other relevant distances, and times and proper times must be fuzzy to enable coherent oscillations to be observed.

Two nontrivial examples were given. Neutrinos propagating in weak fields and in gravitational fields. In both cases the relative phase is modified by the presence of the field. The phase shift is different for a real experiment with measurements in space, and for "gedanken" experiments done in time. We show how the group velocity relates the two descriptions.

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